

# 1 Answers for Practice Test

## 1.1 Laplace Transforms

1(a) After taking LT's and completing the square,  $X(s) = \frac{s+2}{((s+2)^2+1)}$ . Taking ILT,  $x(t) = e^{-2t} \cos(t)$ .

(b)  $x(t) = g(t) * f(t) = \int_0^t e^{-x} \sin(x) f(t-x) dx$ .

(c) Just superimpose (add):  $x(t) = e^{-t} \cos(t) + \int_0^t e^{-x} \sin(x) f(t-x) dx$ .

2(a) T.F. =  $1/(s+2)$ . (b)  $g(t) = \mathcal{L}^{-1}\{1/(s+2)\} = e^{-2t}$ . (c)  $y = g(t) * f(t) = \int_0^t e^{-2(t-x)} f(x) dx$ . (c) Putting  $f(x) = e^{-x} u(x-1)$  and noting that if  $t < 1$ , the integral is 0 since  $u(x-1)$  is 0, then for  $t > 1$ , we get

$$\int_0^1 e^{-2(t-x)} e^{-x} u(x-1) dx + \int_1^t e^{-2(t-x)} e^{-x} u(x-1) dx$$

but again the first integral is 0 and  $u(x-1) = 1$  in the second integral, thus we get  $y(t) = e^{-2t}(e^t - e)u(t-1)$ .

## 1.2 Probability & Statistics

1.(a) Graphing the CDF, we see it has 4 jumps thus the r.v. takes on 4 values with nonzero probability = the height of the jump at that value:

$x$	1	3	5	5.5
$p(x)$	.1	.3	.5	.1

(b)  $Prob(3 \leq X \leq 5.5) = p(3) + p(5) + p(5.5) = .9$ ;  $Prob(X = 3.5) = 0$ .

(c)  $E(X) = \sum_{all\ x} xp(x) = 4.05$  and  $Var(X) = \sum_{all\ x} x^2 p(x) - E(X)^2 = 1.9225$ .

2. (a) T is continuous because F(t) is continuous - note at  $t = 0, 1 - e^{t/1000} = 0$ .

(b)

$$f(t) = F'(t) = \begin{cases} 0 & \text{if } t < 0 \\ \frac{1}{1000} e^{-t/1000} & \text{if } t > 0, \end{cases}$$

(c)  $Prob(T) \geq 500 = 1 - F(500) = .606531$ ,  $Prob(T < 1000) = F(1000) = .632121$  and  $Prob(500 < T < 700) = F(700) - F(500) = .109945$ .

3(a) You can't use the TI92 program, bprob() because you don't know p. But from the binomial, if X = number of defective (success = defective), then  $Prob(X = 100) = p^{100}$ .

(b)  $Prob(\text{at least one nondefective}) = 1 - p^{100}$ . (c) The  $Prob(X \leq 2) = .9999$  either from bprob() or directly computing from binomial formula with  $x=0,1,2$  and adding.

4 (a) Since T is the sum of 100 independent r.v.'s it is approximately normal by the Central Limit Theorem. (b)  $E(T) = 100 \times 1.5 = 150$  min.'s.  $Var(T) = 100 \times 1 = 100$  min<sup>2</sup>. Thus  $T \sim N(150, 100)$ .

(c)  $Prob(T \leq 165) = .933193$  (using zprob() with  $M = 150$ ,  $S = 10$ ,  $X1 = -\infty$ ,  $X2 = 165$ ). (d) To find  $t_0$ , such that  $Prob(T \geq t_0) = .9608$ ,

we standardize, i.e., take  $Z = (T - 165)/10$  and using the normal table together with symmetry get that  $Prob(Z \geq z_0) = .9608$  implies  $z_0 = -1.76$  and thus  $t_0 = 165 + z_0 * 10 = 132.4$  min.'s.

5(a)  $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dx dy = 1$  implies here that

$$\int_{-1}^1 b \int_{-1}^1 kx^2 y^2 dx dy = 4k/9 = 1 \text{ Thus } k = 9/4.$$

(b)  $f_X(x) = \int_{-\infty}^{\infty} f(x, y) dy = \int_{-1}^1 (9/4)x^2 y^2 dy = (3/2)x^2$  if  $-1 \leq x \leq 1$ .

$$\text{Thus } f_X(x) = \begin{cases} (3/2)x^2 & \text{if } -1 \leq x \leq 1 \\ 0 & \text{elsewhere} \end{cases}.$$

$$\text{Similarly, } f_Y(y) = \begin{cases} (3/2)y^2 & \text{if } -1 \leq y \leq 1 \\ 0 & \text{elsewhere} \end{cases}.$$

6. (a) Since we want a 95% 2-sided CI  $1-\alpha = .95$  and so  $z_{\alpha/2} = z_{.025} = 1.96$  - either from tables for the normal or from memory. Thus since  $\sigma = 3.68$ , we get

$$9.44 - (1.96 * 3.68/\sqrt{50}) \leq \mu \leq 9.44 + (1.96 * 3.68/\sqrt{50}) \text{ or } 8.41996 \leq \mu \leq 10.46.$$

(b) For a 1-sided 95% CI,  $z_{\alpha} = z_{.05} = 1.645$  either from tables for the normal or from memory. Thus

$$\mu \leq 9.44 + (1.645 * 3.68/\sqrt{50}) \text{ or } \mu \leq 10.2961. \text{ For 99\% confidence, } 1-\alpha = .99 \text{ and } z_{\alpha/2} = z_{.005} = 2.575 - \text{ either from tables for the normal or from memory. Thus } n = \left(\frac{2.575 * 3.68}{.02}\right)^2 = 22.4486. \text{ So we take } n = 23.$$

### 1.3 Complex Numbers and Taylor Series

1(a)  $(-32)^{1/5} = 32^{1/5} e^{i(\pi+2k\pi)/5}$ ,  $k = 0, 1, 2, 3, 4$ . For  $k = 0$ :  $\frac{\sqrt{5}+1}{2} + \frac{\sqrt{2(5-\sqrt{5})}}{2}i$ ;  $k = 1$ :  $\frac{1-\sqrt{5}}{2} + \frac{\sqrt{2(5+\sqrt{5})}}{2}i$ ;  $k = 2$ :  $-2$ ;  $k = 3$ :  $\frac{1-\sqrt{5}}{2} - \frac{\sqrt{2(5+\sqrt{5})}}{2}i$ ;  $k = 4$ :  $\frac{\sqrt{5}+1}{2} - \frac{\sqrt{2(5-\sqrt{5})}}{2}i$ .

(b)  $(-i)^{1/2} = e^{i(-\pi/2 + 2k\pi)/2}$ ,  $k = 0, 1$ . For  $k = 0$ :  $\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$ ;  $k = 1$ :  $\frac{-\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$ .

(c) In (a) it's a root of  $z^5 + 32 = 0$ , thus complex conjugate pairs. In (b) it's a root of  $z^2 + i = 0$ , thus not necessarily complex conjugate pairs (and they turn out not to be as above shows!).

(d)  $Re(\frac{1+i}{1-i}) = 0$ . (e) Putting  $z=2$  in given polynomial gives  $k = -7/4$ . But then dividing  $z^3 - 7z^2/4 - z + 1$  by  $z - 2$  either by hand or with TI92 gives for the quotient  $\frac{4z^2z-2}{4}$ . But substituting  $z=2$  into this does not give 0 so  $z=2$  is not a double root.

2(a)  $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$ ,  $-1 < x < 1$ , put  $x^2$  in for  $x$  and multiply by  $x$ ,

$$\frac{x}{1-x^2} = x + x^3 + x^5 + \dots = \sum_{n=1}^{\infty} x^{2n-1}$$

with interval of convergence  $-1 < x < 1$ .

(b) Given  $y(1) = 0$ ,  $y'(1) = -1$ , we find  $y''(1) = 0$ , but then  $y''' = xy' + y$ ,  $y'''(1) = -1$ . Continuing differentiating and substituting in  $x=1$ , gives

$$y^{(4)}(1) = -2, y^{(5)}(1) = -1, \text{ so } y(x) = -(x-1) - (x-1)^3/3! - 2(x-1)^4/4! - (x-1)^5/5! + \dots$$

## 1.4 Fourier Analysis

1. (a)  $2L = 2\pi$  so  $L = \pi$ .  $c_0 = 1/(2\pi) \int_{-\pi/2}^{\pi/2} dt = 1/2$ ;  $c_n = 1/(2\pi) \int_{-\pi/2}^{\pi/2} e^{-int} dt = \frac{1}{n\pi} \sin(\frac{n\pi}{2})$ . Thus
- $$f(t) = \frac{1}{2} + \frac{1}{\pi} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{\sin(n\pi/2)}{n} e^{int}.$$
- (b)

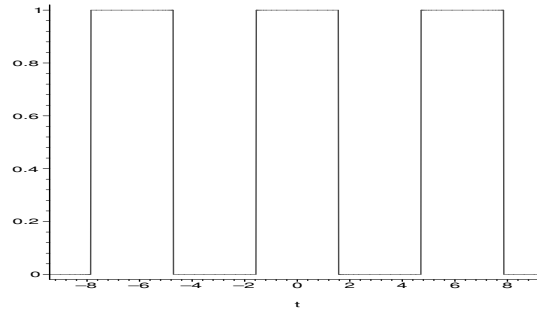


Figure 1: The graph of  $f(t)$ .

- At  $t = \pi$  CFS  $\rightarrow 0$  and at  $t = -3\pi/2$ , CFS  $\rightarrow 1/2$ .
- (c)

$$CFS = \dots + \frac{1}{-3\pi} e^{-3it} + \frac{-1}{-\pi} e^{-it} + 1/2 + \frac{1}{\pi} e^{it} + \frac{-1}{3\pi} e^{3it} + \dots$$

- (d) Note that CFS coefficients are all real so their angles are always either 0 or  $\pi$ .

2 (a)  $y_p(t) = \sum_{n=-\infty}^{\infty} \frac{c_n}{-4n^2 + 4in + 1} e^{2int}$ . (b)  $y_c(t) = Ae^{-t} + Bte^{-t}$ .

- 3(a)  $f(t) = \int_{-\infty}^{\infty} \frac{\sin(2\omega)}{\pi\omega} e^{i\omega t} d\omega = CFI$ . (b) At  $t = 0$ , CFI  $\rightarrow 1$  and at  $t = -1/2$ , CFI  $\rightarrow 1/2$ .

(c)  $\int_0^{\infty} \frac{\sin(2\omega)}{\omega} d\omega = \pi/2$ .

4. (a) Taking FT's,  $G(\omega) = \frac{1}{(i\omega)^2 + 8i\omega + 15}$  since  $\mathcal{F}\{\delta(t)\} = 1$ . But using partial fractions,  $G(\omega) = \frac{1/2}{i\omega + 3} + \frac{-1/2}{i\omega + 5}$ . Thus taking IFT's, we get  $g(t) = (1/2)(e^{-3t} - e^{-5t})u(t)$ .

(b)  $y(t) = g(t) * f(t) = (1/2) \int_{-\infty}^{\infty} (e^{-3x} - e^{-5x})u(x)f(t-x)dx$ .